Function Projective Synchronization between Two Different Chaotic Systems

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A function projective synchronization is defined to synchronize two different systems up to a scaling function matrix f with different initial values. The function projective synchronization is more general than the complete synchronization, the generalized projective synchronization and the modified projective synchronization. The corresponding framework of synchronization is set up and used to achieve a function projective synchronization design of two different chaotic systems: the unified chaotic system and the Rössler system. Feasibility of the proposed control scheme is illustrated through the numerical simulation.

Key words: Generalized Projective Synchronization; Function Projective Synchronization; Rössler System; Unified Chaotic System.

1. Introduction

Research on controlling chaotic systems has seen remarkable growth in a short time span, with "early" studies in the field appearing less than twenty years ago. In 1990, Ott, Grebogy, and Yorke [1] introduced a linear feedback (OGY) method for stabilizing unstable periodic orbits in chaotic systems, which did not require knowledge of the governing equations. The OGY method generated widespread interest, and various modifications and reductions of the scheme quickly followed, in particular, Pyragas [2] presented the time-delay autosynchronization (TDAS) method that has the great advantage of being easily implementable on various experimental systems. Methods for synchronizing chaotic systems developed virtually simultaneously with the developments in chaos control. Pecora and Carroll [3] presented the chaos synchronization method to synchronize two identical chaotic systems with different initial values. Since the pioneering works of these scientists, chaos control and chaos synchronization have received a significant attention in the last few years [4 – 9 and the references therein].

Then different types of synchronization behaviors have been discovered because of potential applications in secure communications. Projective synchronization has been first reported by Mainieri and Rehacek [10] in partially linear systems that the drive and response vectors evolve in a proportional scale – the vectors became proportional. The early projective synchronization is usually observable only in a class of systems with partial-linearity [11]. Then some researchers [12, 13] have achieved control of the projective synchronization in a general class of chaotic systems including non-partially-linear systems, and termed this projective synchronization "generalized projective synchronization" (GPS). Li [14] showed GPS between the Lorenz system and Chen's system. Recently, modification of projective synchronization is proposed by Li in [15] to synchronize two identical systems up to a scaling constant matrix.

In this paper, we propose function projective synchronization (FPS), where the response of the synchronized dynamical states synchronize up to a scaling function matrix f. We synchronize two different chaotic systems (the unified chaotic system [16] and the Rössler system) up to a scaling function matrix. Some numerical simulations are given to show the global synchronization. Here we use the active control method [17] to investigate the function projective synchronization between the unified chaotic system and the Rössler system via u_i ($i = 1, 2, 3$).

The paper is organized as follows: The function projective synchronization of two different chaotic systems is analyzed in Section 2, some numerical simulations are given to verify the effectiveness of our method in Section 3, and finally, some summary and conclusions are given in Section 4.

2. Design of Two Different Chaotic Systems

Recently, based on the previous projective synchronization, we extend the modified projective synchronization proposed by Li [15] and the function projective synchronization, that we propose between two identical chaotic systems, to synchronize two different chaotic systems up to a scaling function matrix. Similarly, the function projective synchronization is characterized by a scaling function matrix:

Definition. Let $\dot{x} = F(x,t)$ is the drive chaotic system, $\dot{y} = G(y,t) + U$ is the response system, where $x = (x_1(t), x_2(t), \ldots, x_m(t))^T$, $y = (y_1(t), y_2(t), \ldots, y_m(t))^T$, and $U = (u_1(x,y), u_2(x,y), \ldots, u_m(x,y))^T$ is a controller to be determined later. Denote $e_i = x_i - f_i(x)y_i$ $(i = 1,2,\ldots,m)$, $f_i(x)$ $(i = 1,2,\ldots,m)$ are functions of x. If $\lim_{t\to\infty} \|e\| = 0$, $e = (e_1, e_2, \ldots, e_m)$, we call these two different chaotic systems "function projective synchronization (FPS)", and we call f a "scaling function matrix".

Consider the drive system in the form

$$\dot{x} = A_1 x + h_1(x,t). \tag{1}$$

Assume that the response system is

$$\dot{y} = A_2 y + h_2(y,t) + U, \tag{2}$$

where $x, y \in R^n$, A_1, A_2 are $m \times m$ constant matrixes, $h_1, h_2 : R^m \to R^m$ are nonlinear function vectors, and U is a controller to be determined later.

Theorem. For an invertible diagonal function matrix f, function projective synchronization between the two systems (1) and (2) will occur, if the following conditions are satisfied:

(i) $U = f^{-1}h_1(x,t) + (f^{-1}A_1 f - A_2)y + f^{-1}B(x - fy) - h_2(y,t) - f^{-1}gy$, where $g = \mathrm{diag}(\dot{f}_1, \dot{f}_2, \ldots, \dot{f}_m)$, and $B \in R^{m\times m}$.

(ii) The real parts of all the eigenvalues of $(A_1 - B)$ are negative.

Proof. From $e = x - fy$ in the definition of FPS, one can get

$$\begin{aligned} \dot{e} &= \dot{x} - f\dot{y} - gy \\ &= A_1 x + h_1(x,t) - f(A_2 y + h_2(y,t) + U) - gy \\ &= A_1 x + h_1(x,t) - fA_2 y - fh_2(y,t) - gy \\ &\qquad - h_1(x,t) - A_1 fy + fA_2 y \\ &\qquad - B(x - fy) + fh_2(y,t) + gy \\ &= (A_1 - B)e. \end{aligned} \tag{3}$$

With regards the Lyapnov stability theory and for a feasible control, the feedback B must be selected such that all the eigenvalues of $(A_1 - B)$ have negative real parts. Thus, if the controllability matrix $(A_1 - B)$ is in full rank, the system (3) is asymptotically stable at the origin, which implies that (1) and (2) are in the state of function projective synchronization.

In this letter, the active control method [17] is adopted to obtain the gain matrix B for any specified eigenvalues of $(A_1 - B)$.

It is necessary to point out that the scaling function matrix f has also no effect on the eigenvalues of $(A_1 - B)$ like the modified projective synchronization. Thus one can adjust the scaling matrix arbitrarily during control without worrying about the control robustness. The FPS is more general: When $f_1 = f_2 = \ldots = f_m = 1$, $f_1 = f_2 = \ldots = f_m = \alpha$ and $f_1 = \alpha_1, f_2 = \alpha_2, \ldots, f_m = \alpha_m$, the complete synchronization, the projective synchronization and the modified projective synchronization will appear, respectively.

3. FPS between the Unified Chaotic System and the Rössler System

The drive system – the unified chaotic system – is presented as follows:

$$\begin{aligned} \dot{x_1} &= (25\alpha + 10)(x_2 - x_1), \\ \dot{x_2} &= (28 - 35\alpha)x_1 - x_1 x_3 + (29\alpha - 1)x_2, \\ \dot{x_3} &= x_1 x_2 - \frac{8+\alpha}{3} x_3, \end{aligned} \tag{4}$$

and the response system – the Rössler system – is given by

$$\begin{aligned} \dot{y_1} &= -y_2 - y_3 + u_1, \\ \dot{y_2} &= y_1 + a y_2 + u_2, \\ \dot{y_3} &= b + y_1 y_3 - c y_3 + u_3. \end{aligned} \tag{5}$$

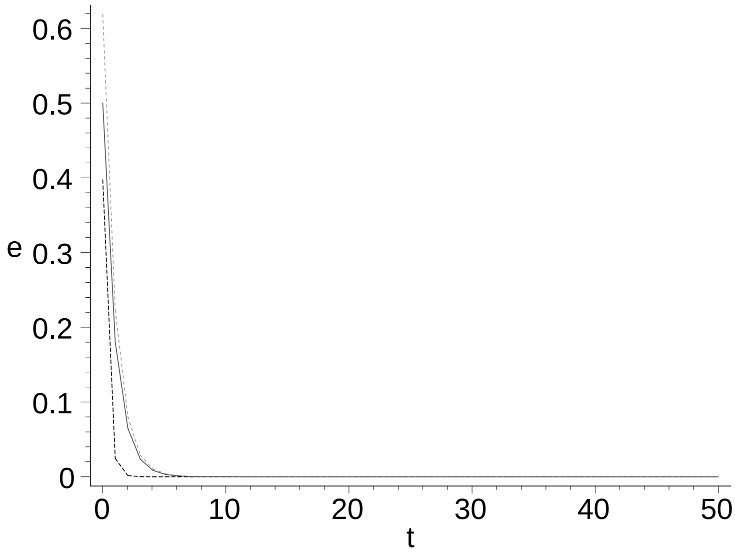

Fig. 1. Time response of the synchronization error e. "$-$" denotes for the orbit of e_1, "$-.-$" for the orbit of e_2, and "$--$" for the orbit of e_3.

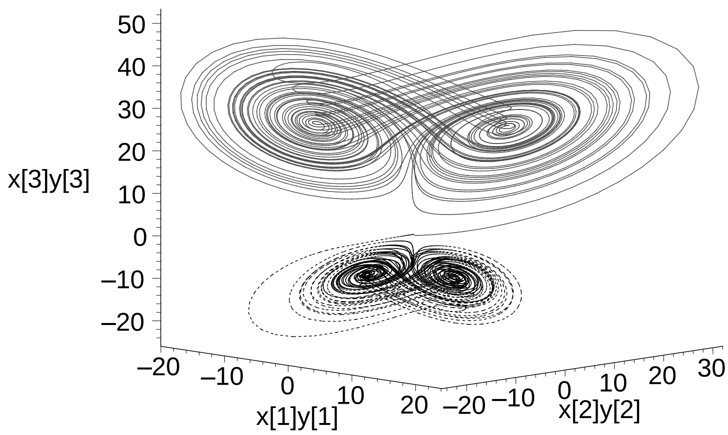

Fig. 2. "$-$" denotes for the drive system orbit and "$--$" for the response system orbit.

In [18], Yan explored the two chaotic systems based on the backstepping design method and Lyapunov theory. Here by means of the active control method [17] and feedback stepping method, we can choose $f(x) = \mathrm{diag}(m_1 - m_2, m_1 - m_3\tanh(x_3), m_1 - m_3\tanh(x_3))$. We set

$$B = \begin{pmatrix} -(25\alpha+9) & 25\alpha+10 & 0 \\ 28-25\alpha & 29\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is easy to see that the controllability matrix

$$A_1 - B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -\frac{8+\alpha}{3} \end{pmatrix}$$

has all eigenvalues with negative real parts. And therefore the three control functions u_i $(i = 1,2,3)$ are in the form

$$\begin{aligned} u_1 &= \frac{(25\alpha+10)x_2 - (25\alpha+9)x_1}{m_1 - m_2} - y_1 + y_2 + y_3, \\ u_2 &= \frac{(28-35\alpha)x_1 - x_1x_3 + 29\alpha x_2 + m_3y_2\tanh^2(x_3)}{m_1 - m_3\tanh(x_3)} \\ &\quad - y_1 - (a+1)y_2, \\ u_3 &= \frac{x_1x_2 + m_3y_3\tanh^2(x_3)}{m_1 - m_3\tanh(x_3)} - b - y_1y_3 + cy_3 \\ &\quad - \tfrac{1}{3}(8+\alpha)y_3. \end{aligned} \tag{6}$$

Then we arbitrarily give the initial states of the two different chaotic systems – the unified chaotic system and the Rössler system – (0.1, 0.5, 0.2) and (0.2, 0.3, 0.5), respectively, and choose $\alpha = 0.2$, $m_1 = 0$, $m_2 = 2$, $m_3 = 2$. Therefore, the initial values of e are (0.5, 0.6184251921, 0.3973753202). Figure 1 displays the

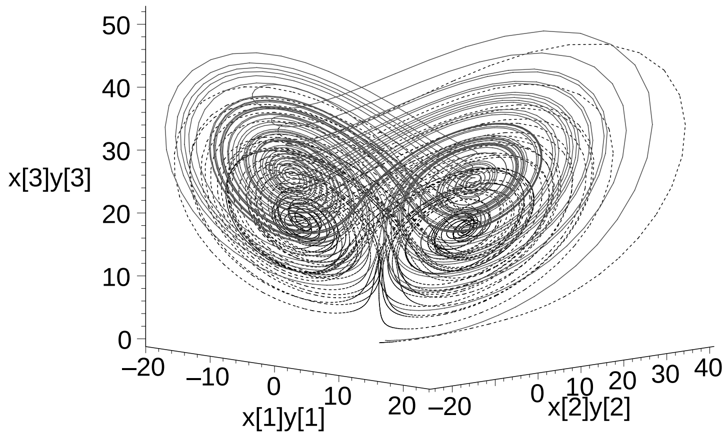

Fig. 3. "−" denotes for the drive system orbit and "−−" for the orbit of the response system multiplied by the scaling function matrix f.

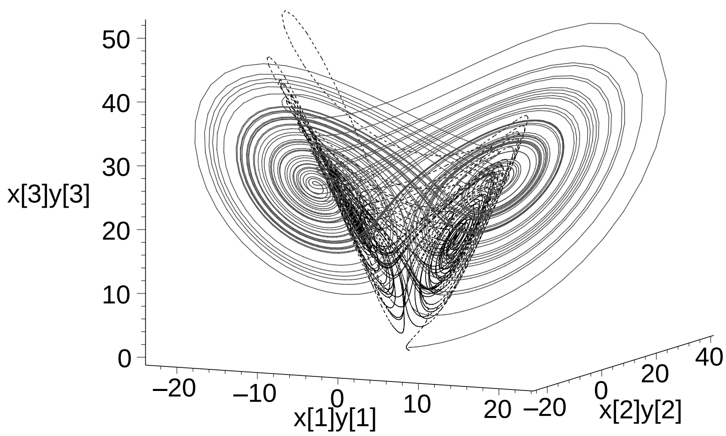

Fig. 4. "−" denotes for the drive system orbit in Fig. 3 and "−−" for the orbit of the response system in Fig. 3 with "$y_1 \to -y_1$".

time response of the synchronization error e at the initial states above. Obviously, e converges to zero finally after the controller is activated. This means, all the state variables tend to be synchronized. From Fig. 2, we can easily observe that the ratio of the amplitudes of the two systems tends to the scaling function matrix f. A natural problem is whether the two different chaotic systems completely synchronize if we make the response system multiplied by the scaling function matrix f. Working under this idea, we can get the result in Figure 3. In order to see clearly, we take $y_1 \to -y_1$, which is presented in Figure 4.

One can properly choose the values of α, f to get another better result. These figures above fully reveal our results.

4. Summary and Conclusions

In summary, the more general definition of synchronization, named function projective synchronization, is presented. Based on the symbolic computation and active control method, we set up the corresponding scheme of function projective synchronization and show how to synchronize two different chaotic systems, the unified chaotic system and the Rössler system, and make them globally synchronized in one coordinate with a scaling function matrix f. Numerical experiments show that the method we set up works well, and that it can be used for other chaotic systems and hyperchaotic systems.

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